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NEW GEOMETRIC FIXED POINT THEOREMS

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Abstract. In this paper it is proved the following main result that if T is a self-map on a complete metric space (X, ρ) and if there exists an upper semicontinuous bounded above function $G : X \to \mathbf{R}$ such that

$$\text{(A)} \qquad \rho[x, Tx] \leq G(Tx) - G(x)$$

for every $x \in X$, then T has a fixed point in X. This paper presents and some other results of this type.

1. Introduction and results

The notion of order, and the notion of completeness, have each led to a fixed point statement. We now obtain geometric results of fixed points based on an interplay of these two notions.

In recent years a great number of papers have presented considerations of the well-known Caristi's theorem, which is equivalent to Ekeland's minimization theorem.

This paper continues the study of the preceding results based on a new geometry of a condition for fixed points.

Theorem 1. *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exists a bounded above function $G : X \to \mathbf{R}$ such that*

$$\text{(A')} \qquad \rho[x, Tx] \leq G(Tx) - G(x)$$

for every $x \in X$. If $x \mapsto \rho[x, Tx]$ is a lower semicontinuous function, then T has a fixed point $\xi \in X$ and $T^n x \to \xi (n \to \infty)$ for each $x \in X$.

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Proof. Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{T^n x\}_{n\in\mathbf{N}}$ is a Cauchy sequence. Let n and m $(n < m)$ be any positive integers. From (A') we have

$$\sum_{i=0}^{n} \rho[T^i x, T^{i+1}x] \leq G(T^{n+1}x) - G(x) ,$$

and thus, since G is a bounded above functional, we obtain the following fact:

$$\rho[T^n x, T^m x] \leq \sum_{i=n}^{m-1} \rho[T^i x, T^{i+1}x] \to 0 \ \ (m,n \to 0) .$$

Hence $\{T^n x\}_{n\in\mathbf{N}}$ is a Cauchy sequence in X and, by completeness, there is $\xi \in X$ such that $T^n x \to \xi$ $(n \to \infty)$. Since $x \mapsto \rho[x, Tx]$ is a lower semicontinuous function at ξ,

$$\rho[\xi, T\xi] \leq \lim . \inf \rho[T^n x, T^{n+1}x] = 0 .$$

Thus $T\xi = \xi$, and we have shown that for each $x \in X$ the sequence $\{T^n x\}_{n\in\mathbf{N}}$ converges to a fixed point of T. This completes the proof.

As an immediate application of the preceding statement, as a directly consequence, we obtain the following fact.

Theorem 1a. *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exist a bounded above function $G : X \to \mathbf{R}$ and an arbitrary fixed integer $k \geq 0$ such that*

$$\rho[x, Tx] \leq G(Tx) - G(x) + \cdots + G(T^{2k+1}x) - G(T^{2k}x)$$

and $G(T^{2i}x) \leq G(T^{2i+1}x)$ for $i = 0, 1, \ldots, k$ and for every $x \in X$. If $x \mapsto \rho[x, Tx]$ is lower semicontinuous, then T has a fixed point $\xi \in X$.

We remark that the existence of a fixed point for a contractive map T in a complete metric space (X, ρ) is a consequence of Theorem 1; for if $\rho[Tx, Ty] \leq \alpha\rho[x, y]$ with $0 \leq \alpha < 1$, we have $\rho[Tx, T^2x] \leq \alpha\rho[x, Tx]$, therefore

$$\rho[x, Tx] - \alpha\rho[x, Tx] \leq \rho[x, Tx] - \rho[Tx, T^2x]$$

so, with the function $G(x) := (\alpha - 1)^{-1}\rho[x, Tx]$, the conditions of Theorem 1 are satisfied.

We notice that the proof of Theorem 1 is given in a form without Axiom of Choice. But, the following variant of Theorem 1 we give via Zorn's lemma in the following form.

Theorem 2. *Let T be a self-map on a complete metric space (X,ρ). Suppose that there exists an upper semicontinuous bounded above function $G: X \to \mathbf{R}$ such that*

$$\text{(A)} \qquad \rho[x,Tx] \leq G(Tx) - G(x)$$

for every $x \in X$. Then T has a fixed point in X.

A part proof for this statement is analogous to the proof of Theorem 1. A brief proof of this statement based on the preceding facts and D-Ordering Principle (dually form) may be found in Tasković [6].

Proof of Theorem 2. (*Application of Zorn's lemma*). Define a relation $\preceq_{G,\rho}$ on X by the folowing condition:

$$a \preceq_{G,\rho} b \text{ if and only if } \rho[a,b] \leq G(b) - G(a) \,.$$

It is to verify that $\preceq_{G,\rho}$ is a partial ordering (asymmetric and transitive relation) in X. The space X together with this partial ordering is denoted by $X_{G,\rho}$.

Fix $t \in X$ and use Zorn's lemma to obtain a maximal (relative to set inclusion) chain M of $X_{G,\rho}$ containing t. Let $M := \{x_\alpha\}_{\alpha\in I}$ and $x_\alpha \preceq_{G,\rho} x_\beta$ if and only if $\alpha \leq \beta$ $(\alpha,\beta \in I)$, where I is totally ordered.

Now $\{G(x_\alpha)\}_{\alpha\in I}$ is an increasing net bounded above in $\mathbf{R}$, so there exists $r \in \mathbf{R}$ such that $G(x_\alpha) \to r$ as $\alpha \uparrow \infty$. Thus, as in the proof of Theorem 1, we obtain that $\{x_\alpha\}_{\alpha\in I}$ is a Cauchy net in X.

By completeness there is $x \in X$ such that $x_\alpha \to x$ as $\alpha \uparrow \infty$. Since G is upper semicontinuous we obtain that is $\lim.\sup G(x_\alpha) \leq G(x)$. Also, for $\alpha \leq \beta$,

$$\rho[x_\alpha, x_\beta] \leq G(x_\beta) - G(x_\alpha) \,,$$

and, letting $\beta \uparrow \infty$, $\rho[x_\alpha, x] \leq G(x) - G(x_\alpha)$ yielding $x_\alpha \preceq_{G,\rho} x$ for $\alpha \in I$. Since M is a maximal chain, we have $x \in M$. On the other hand, also, (A) holds so it follows that

$$x_\alpha \preceq_{G,\rho} x \preceq_{G,\rho} Tx \quad \text{for } \alpha \in I \,,$$

and, by maximality, $Tx \in M$. Therefore $Tx \preceq_{G,\rho} x$ and it follows that $Tx = x$. The proof is complete.

As an immediate application of D-Ordering Principle (dually form) we obtain the following directly generalization of Theorem 2.

Theorem 2a. *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exist an upper semicontinuous bounded above function $G : X \to \mathbf{R}$ and an arbitrary fixed integer $k \geq 0$ such that*

$$\rho[x, Tx] \leq G(Tx) - G(x) + \ldots + G(T^{2k+1}x) - G(T^{2k}x)$$

and $G(T^{2i}x) \leq G(T^{2i+1}x)$ for $i = 0, 1, \ldots, k$ and for every $x \in X$. Then T has a fixed point in X.

An explicit suitable proof of this statement (based on the D-Ordering Principle, dually form) may be found in Tasković [6].

In connection with the preceding, in 1975 J. Caristi proved the following important result in nonlinear functional analysis (see: Browder [1]).

Theorem 3. (Caristi [2], Kirk [4]). *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exists a lower semicontinuous function $G : X \to \mathbf{R}_+^0 := [0, +\infty)$ such that*

$$\text{(CK)} \qquad \rho[x, Tx] \leq G(x) - G(Tx)$$

for every $x \in X$. Then T has a fixed point in X.

Some remarks. We notice that the inequality (A) is not dually, in comparable, with the inequality (CK). Thus implies that, Theorem 2 is not dually result of Theorem 3, of course. This mean that Theorem 2 (as and Theorem 1) is a totally new result in the geometric fixed point theory.

Otherwise, a variant of Theorem 3 (without the lower semicontinuity for the functional $G : X \to \mathbf{R}_+^0$) may be found in Tasković [5].

2. Two open problems

Problem 1. Let T be a mapping of a complete metric space (X, d) into itself. Suppose that there exist a bounded above function $G : X \to \mathbf{R}$, a metric $d_p : X \times X \times \mathbf{R} \to \mathbf{R}$ and an arbitrary fixed integer $k \geq 0$ such that

$$d_p(x, Tx) \leq G(Tx) - G(x) + \cdots + G(T^{2k+1}x) - G(T^{2k}x)$$

and $G(T^{2i}x) \leq G(T^{2i+1}x)$ for $i = 0, 1, \ldots, k$ and for every $x \in X$. If G is an upper semicontinous function or $x \mapsto d_p(x, Tx)$ is a lower semicontinous function, does T have a fixed point in the metric space X?

Problem 2. We notice that the preceding proof of Theorem 2 is given via Zorn's lemma. Does a new proof of Theorem 2 can be given elementary without Axiom of Choice?

Some remarks. We notice that the preceding statements we can modify in the following sence. Naimely, the next statement follows from Theorem 1 as follows.

Theorem 1b. *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exists a bounded above function $G : X \to \mathbf{R}$ such that for any $x \in X$, with $x \neq Tx$, there exists $y \in X \backslash \{x\}$ with property*

$$\text{(B)} \qquad \rho[x, y] \leq G(y) - G(x) .$$

If $x \mapsto \rho[x, Tx]$ is a lower semicontinuous function, then T has a fixed point in X.

On the other hand, as an immediately consequence of Theorem 2, we obtain the following fact as follows.

Theorem 2b. *Let T be a self-map on a complete metric space (X, ρ). Suppose that there exists an upper semicontinuous bounded above function $G : X \to \mathbf{R}$ such that for any $x \in X$, with $x \neq Tx$, there exists $y \in X \backslash \{x\}$ with property (B). Then T has a fixed point in X.*

A brief suitable proof of this statement based on Zorn' s lemma may be found in Tasković [5].

3. References

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